Physics 216/116

Lecture V — Instabilities I

So far: — basic eqns.
— Potential Flow
— Low Re Flow

Sphere + potential Flow

$\longleftrightarrow$ General ideas wake

S + Stokesian flow

$\downarrow$ coming

P + Blasius Boundary Layer (Laminar)

$\downarrow$ $\leftarrow$ wakes, drag, lift

Turbulent wakes, Turbulence

all: { energy source for flow is body motion (u) }

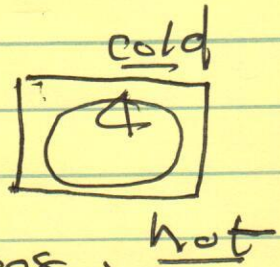
oo: Instability → { continuation → KH ; diversion → RT, RB }

[stored free energy → fluid motions → chaos, turbulence dissipation.]

⇒ Relaxation — critical stage usually is linear instability

① — stored free energy { eqbrm + small pert → growth }

i.e.

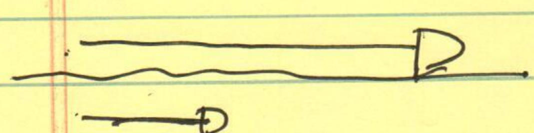


Rayleigh–Benard convection thermal

→ $\nabla T \Leftrightarrow$ buoyancy energy

⇒ rises hot air

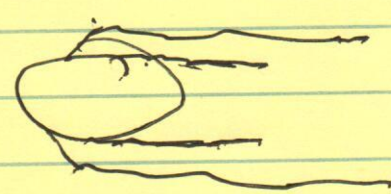
② or



Kelvin–Helmholtz, shear flow

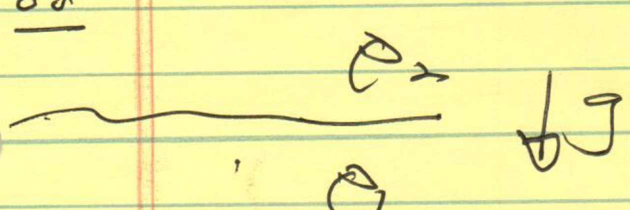
→ $\nabla V \Leftrightarrow$ kinetic energy flow shear

relevant to breakdown of wake (after separation)



⇒ onset of turbulent wake

③ or



Rayleigh–Taylor

→ $\nabla\rho + g$ (buoyancy)

but, heat not central.

→ gravitational potential energy.

Real Story / Question:

Relaxation $\Rightarrow$ Linear Instability $\rightarrow$ Nonlinear Evolution saturation

$\Rightarrow$ Final state

{ (incl. dissipation)
{ often turbulent.

Hydro stability is "huge" subject

c.f. Chandrasekhar

[Theory of Hydrodynamic and Hydromagnetic Stability].

Here: — First step.

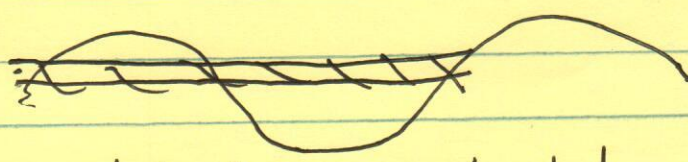
— 2 classes $\rightarrow$
① interfacial instabilities
$\Rightarrow$ RT, KH
② Convection
$\Rightarrow$ RB.

+ homework

1.) Interfacial Instabilities

if L st $\quad 1/L = \left(\frac{1}{\rho}\frac{\partial \rho}{\partial z}\right), \ \frac{1}{V}\frac{\partial V}{\partial z}$

$k_x L \ll 1$



$\Rightarrow$ treat gradient as held in an interface / layer.

⇒ strategy: – 2 homogeneous media

⊕

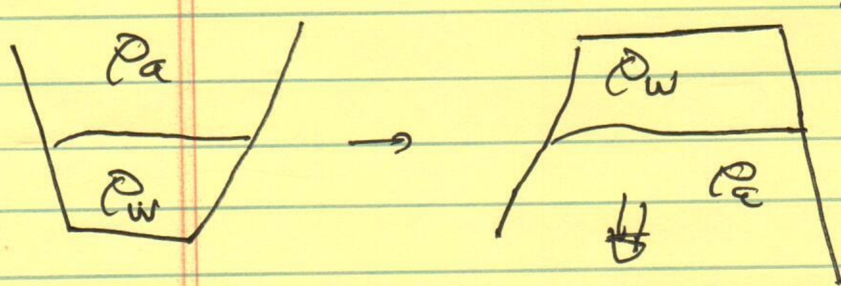
– matching conditions

⟺ significant overlap with theory of { surface phenomena, droplets, etc.

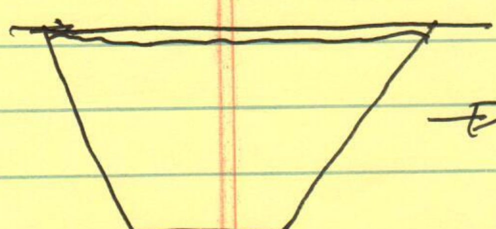
⇒ biophysics:

n.b. in 'Life at low Reynolds number,' surface tension relevant.

c) Prime Example 1: Rayleigh–Taylor (c.f. posted papers, especially Taylor 1950).



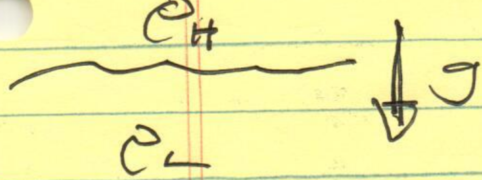
why? ⇒ Ripples on surface grow ⇒ R-T instability



→ invert

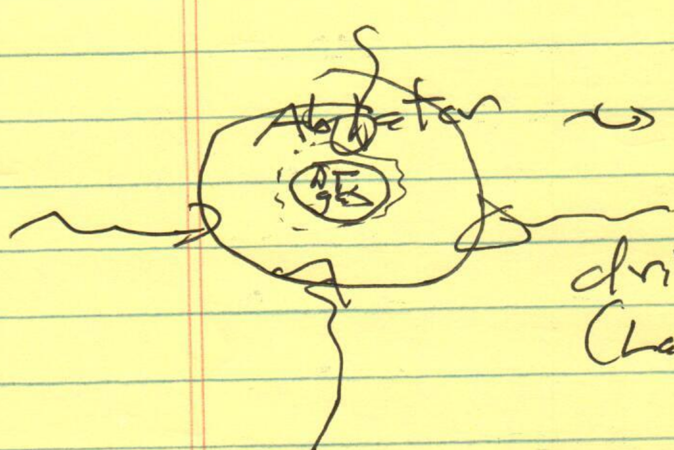
nothing happens!

⇒ cardboard effectively takes $\sigma_{surf.} \to \infty$.



→ contrived ?!

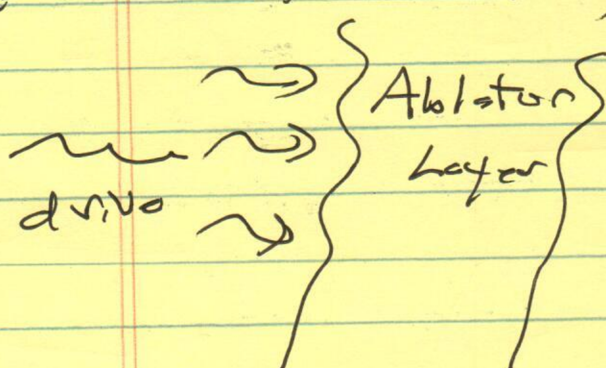
<u>but</u> ICF (controlled and otherwise)



→ Implosion

⇒ ablation-driven rocket:

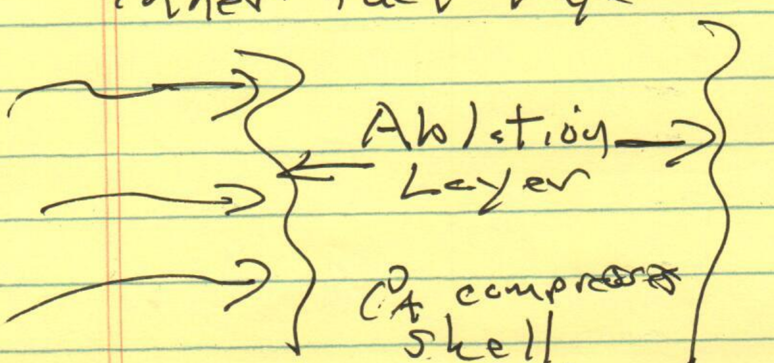
driver conclusions



Fuel shell.

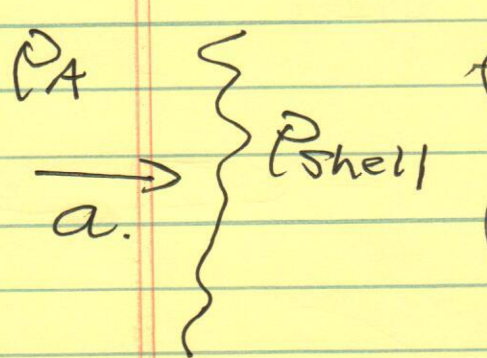
treat: compression dynamics as rocket equation

⇒ drive causes ablator layer to heat and expand, thus compressing inner fuel layer



Consider situation:

$\rho_{shell} > \rho_a$.

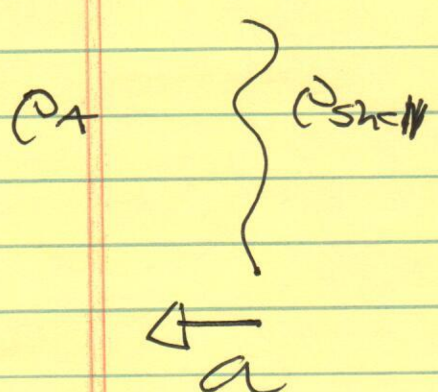


$\boxed{\text{"Light/low density fluid pushing on (gas) (i.e. accelerating onto) heavy fluid."}}$

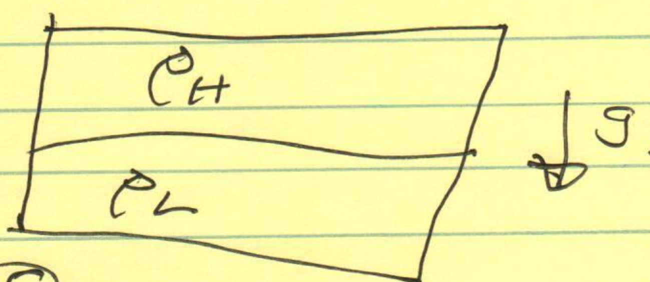
i.e.

c.f. Taylor's paper.

in frame of ablator:

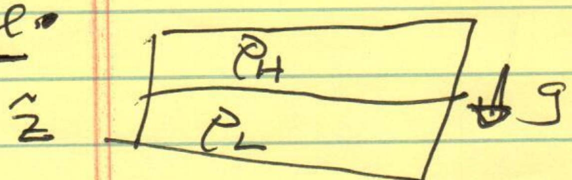


$\Rightarrow$ equivalent to



Both cases: $\nabla P \, \nabla \rho < 0$

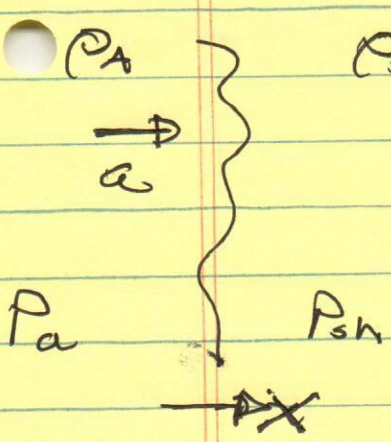
i.e.



$\nabla \rho > 0$

$\nabla P < 0$.

$\underline{\nabla P = -\rho g}$



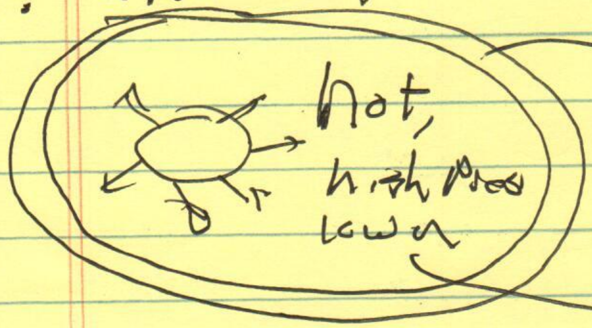
$\vec{\nabla}_x \rho$ [illegible] 0

$\nabla_x P < 0$.

$P_a > P_{shell}$

i.e. both $\nabla P \nabla \rho < 0$

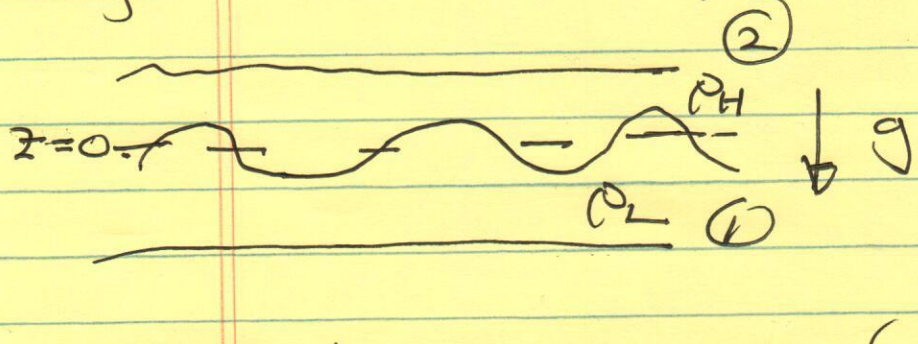
n.b.: also supernovas:

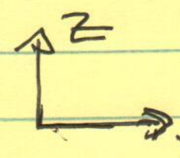


→ colder, high density material

→ hot, low density high pressure

so, hereafter: simple case





— $\nabla\cdot\underline{v} = 0$ i.e. ($v \ll c_s$ etc.)

— ideal fluid (add visc. on HW)

Equilibrium

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v} = -\frac{\underline{\nabla} p}{\rho} + \rho\,\underline{g}$$

$\underline{\rho \text{ const.}}$

$\underline{v} \to 0$

$$\nabla^2 p = 0$$

$\to$ vert,
$$\partial_z^2 p = 0$$
$$p = p_0 + p' z$$

but $\underline{dp/dz = -\rho g}$

$$p_2' = -\rho_2 g$$

$$p_1' = -\rho_1 g$$

$\Delta p > 0$

going $\downarrow$

as interface ($k_x L_z \ll 1$), vorticity localized in interface. So treat fluid as irrotational

$\underline{\omega} = 0$, $\quad \underline{v} = \underline{\nabla}\phi$

$\underline{\nabla}\cdot\underline{v} = 0 \qquad \nabla^2\phi = 0$

e^{-kz}

— — — $z=0$

e^{+kz}

$$\phi = \sum_k e^{ikx}\phi_k(z)$$

$$\left(\partial^2/\partial z^2 - k^2\right)\phi_k(z) = 0$$

$$\phi_k = \begin{cases} e^{-kz} & z>0 \\ e^{kz} & z<0 \end{cases} \qquad (k>0)$$

at interface ($z=0$) = matching conditions

→ pressure balance across interface $\qquad P(0_+) = P(0_-)$

(else interface in motion on acoustic time scales)

$$V_z(0_+) = V_z(0_-)$$

$$\left.\frac{\partial \phi}{\partial z}\right|_{2,\; z\to 0} = \left.\frac{\partial \phi}{\partial z}\right|_{1,\; z\to 0}$$

i.e.

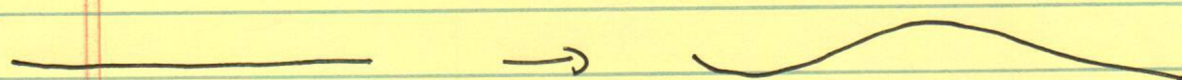
$$\int_{0_-}^{0_+} \left[\frac{\partial^2 \phi}{\partial z^2} - k^2 \phi\right] = 0$$

Note: V_z b.c. immediately forces

$$-k\phi_2 = k\phi_1 \qquad \Rightarrow \phi_2 = -\phi_1$$

What of dynamics?

— interface ripples



$z=0$

$z_i = 0 + \eta(x,t)$

— displacement of interface.

— η specifies interface position

Note: $\phi = \phi(x, z_i, t)$ $\left\{\text{at interface position}\right.$

$= \phi(x, 0+\eta, t)$

linear theory

$\cong \phi(x, 0, t)$

i.e. linear theory $\left\{ \begin{array}{l} \phi(x, z_i, t) \to \phi(x, 0, t) \\ k\eta \ll 1 \end{array} \right.$

Now must account for force of gravity with displaced interface in Bernoulli equation:

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right) = -\underline{\nabla}p - \rho\underline{g}$$

$$\rho\left(\frac{\partial}{\partial t}\underline{v} + \underline{\nabla}\left(\frac{v^2}{2}\right) - \cancel{\underline{v}\times\underline{\omega}}\right) = -\underline{\nabla}p - \rho g\hat{z}$$

$(g > 0)$

$\underline{v} = \underline{\nabla}\phi$, $v_z = \partial_z\phi$

$\int_0^\eta dz\, v_{z_1} = \phi_1$, $\int_\eta^0 dz\, v_{z_2} = \phi_2$

so

$$\boxed{\frac{\partial\phi}{\partial t} + \frac{(\underline{\nabla}\phi)^2}{2} = -\frac{p}{\rho} - g\eta}$$

[absent gravity, $p = -\rho\, \partial\phi/\partial t$

so

$$p = -\rho g z - \rho \frac{\partial \phi}{\partial t}$$

and @ interface

$$p = -\rho g \eta - \rho \frac{\partial \phi}{\partial t}$$]

and finally have equation / dynamic boundary condition for displacement:

$$\boxed{\frac{d\eta}{dt} = \partial_z \phi \Big|_0 = v_z}$$

$\Rightarrow$

$$\frac{\partial \eta}{\partial t} + \underbrace{\underline{v}}_{\nabla\phi} \cdot \underline{\nabla}\, \eta = \frac{\partial \phi}{\partial z}\Big|_0$$

For stability : linearize

$$\frac{\partial \tilde{\phi}}{\partial t} = -\frac{\tilde{p}}{\rho} - g\eta$$

$$\frac{\partial \tilde{\eta}}{\partial t} = \frac{\partial \tilde{\phi}}{\partial z}\Big|_0 .$$

So noting $\rho_2 \neq \rho_1$

$$\tilde{p}^{(1)}\Big|_0 = \tilde{p}^{(2)}\Big|_0$$

$$\tilde{\phi}^{(1)}\Big|_0 = -\phi^{(2)}\Big|_0$$

$$\rho_2 \frac{\partial \tilde{\phi}^{(2)}}{\partial t} + g\rho_2 \tilde{\eta} = \rho_1 \frac{\partial \tilde{\phi}^{(1)}}{\partial t} + g\rho_1 \tilde{\eta}$$

$$g(\rho_2 - \rho_1)\tilde{\eta} = \rho_1 \frac{\partial \tilde{\phi}^{(1)}}{\partial t} - \rho_2 \frac{\partial \tilde{\phi}^{(2)}}{\partial t}$$

$$= (\rho_1 + \rho_2)\frac{\partial \tilde{\phi}^{(1)}}{\partial t}$$

$$\boxed{\begin{aligned} \frac{\partial \tilde{\phi}^{(1)}}{\partial t} &= g\left[\frac{(\rho_2 - \rho_1)}{(\rho_2 + \rho_1)}\right]\tilde{\eta} \\ \frac{\partial \tilde{\eta}}{\partial t} &= \frac{\partial \tilde{\phi}^{(1)}}{\partial z}. \end{aligned}}$$

$$\Rightarrow \boxed{\frac{\partial^2 \tilde{\phi}^{(1)}}{\partial t^2} = g\left[\frac{(\rho_2 - \rho_1)}{(\rho_1 + \rho_2)}\right]\frac{\partial \tilde{\phi}^{(1)}}{\partial z}.}$$

using $\phi \sim e^{-i\omega t} e^{kz} e^{ikx}$

$$\Rightarrow \quad -\omega^2 = \left[g(\rho_2 - \rho_1) / (\rho_1 + \rho_2) \right] k.$$

$\rho_2 = \rho_H$
$\rho_2 = \rho_L$

$$\boxed{\gamma^2 = gAk \;, \quad A = \frac{\rho_H - \rho_L}{\rho_H + \rho_L}}$$

– free energy
– inertia

Atwood #

i.) $\rho_H = H_2O$ $\qquad \lambda \sim 1$ cm.
$\rho_L =$ air $\qquad \tau_g \sim .1$ sec.
(fast!)

ii.) $\rho_2 =$ air $\qquad \rho_{air}/\rho_{H_2O} \rightarrow 0$
$\rho_1 =$ water

$$\omega = \sqrt{kg}$$

→ dispersion relation for surface gravity wave.

(stable wave counterpart of R-T)

iii) $\gamma \sim (gAk)^{1/2}$

→ shorter wavelengths grow faster?!

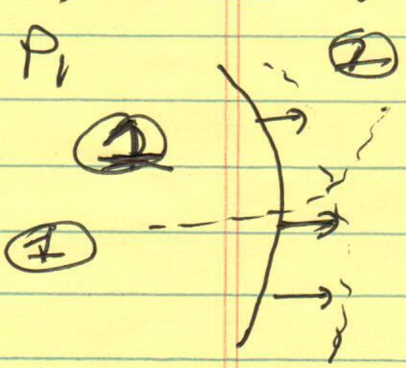
⇒ small scale effects?
cut-off, regulator?

— viscosity (HW)

— surface tension (*)

— finite layer width $\left(k_y L_z \gtrsim 1 \right)$

Surface Tension → force due increase in surface area interface



① expands

↳ isothermal displacement

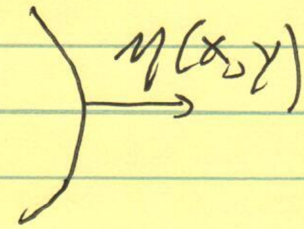
$$dF = -p_1\, dV - p_2(-dV) + \sigma\, dA$$

dF: change in free energy

$-p_1\, dV$: ① expands into ②

$\sigma\, dA$: change in surface area of interface

$$dV = dA\, \underbrace{d\eta}_{\text{displacement}}$$



so

$$dA = \int dx\,dy \left(1+\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2\right)^{1/2} - \int dx\,dy$$

small displacement (slope):

$$\cong \int dx\,dy \left(1+\frac{1}{2}\left(\frac{\partial\eta}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial\eta}{\partial y}\right)^2\right) - \int dx\,dy$$

$$= \int dx \int dy\ \frac{1}{2}\left[\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2\right]$$

IBP

$$= \int dx \int dy \left(\underbrace{-\nabla^2\eta}_{\text{curvature of surface displacement}}\right)\delta\eta$$

so

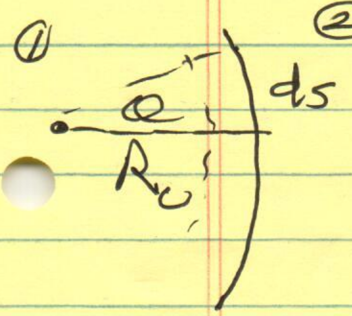
$$dF = \int\left[(\rho_2-\rho_1)\,dA_0 - \sigma\nabla^2\eta\, dA_0\right]\delta\eta$$

so criterion for equilibrium:

$$\boxed{P_2 - P_1 = \sigma \nabla^2 \eta}$$

More generally: $dF = (P_2 - P_1)\, dA_0\, d\eta + \sigma\, dA$

Now consider arbitrary (i.e. not "weakly curved" interface):



$$ds = (R_0 + d\eta)\, d\theta$$

$$\simeq dl_0 \left(1 + d\eta / R_1\right)$$

radius curv

In general, surface parametrized by $\underline{\underline{2}}$ radii of curvature, R_1, R_2:

$$dA = \int dl_1\, dl_2 \left(1 + \frac{d\eta}{R_1}\right)\left(1 + \frac{d\eta}{R_2}\right) - \int dl_1\, dl_2$$

$$= \int dl_1\, dl_2 \left(1/R_1 + 1/R_2 \right) d\eta$$

so

$$dF = \int \left[(P_2 - P_1) + \sigma \left(1/R_1 + 1/R_2 \right) \right] dA_0\, d\eta$$

so, for equilibrium with interface (general)

$$\boxed{\sigma\left(1/R_1 + 1/R_2\right) = P_1 - P_2}$$

[Laplace's Law

- Given 2-phase equilibrium (separate domains), can use Laplace Law to estimate droplet size for immiscible liquids

i.e. $P_1 > P_2 \Rightarrow R \sim \sigma/(P_1 - P_2)$

Now, back to R-T, S-W:

$\rho_H \gg \rho_L$

$\rho_H + \rho_L \to \rho_H$

$$P \to P - \rho\gamma_T \nabla_h^2 \eta$$

$\gamma_T \equiv \sigma/\rho$. $\rightarrow \sigma$ for each interface i.e. water-air, etc.

$\Rightarrow$

$$\gamma_{R-T} = \left(kgA^{1} - \gamma_T k^3\right)^{1/2}$$

cut-off

$$k_{max}\Big|_{unst} \sim \left(g/\gamma_T\right)^{1/2}$$

$\rightarrow$ limits range of unstable modes.

For stable case:

$$\left.\omega^2 = gk + \frac{\sigma}{\rho} k k^2 \right\} \text{gravity-capillary}$$

$\downarrow$ gravity wave (long)

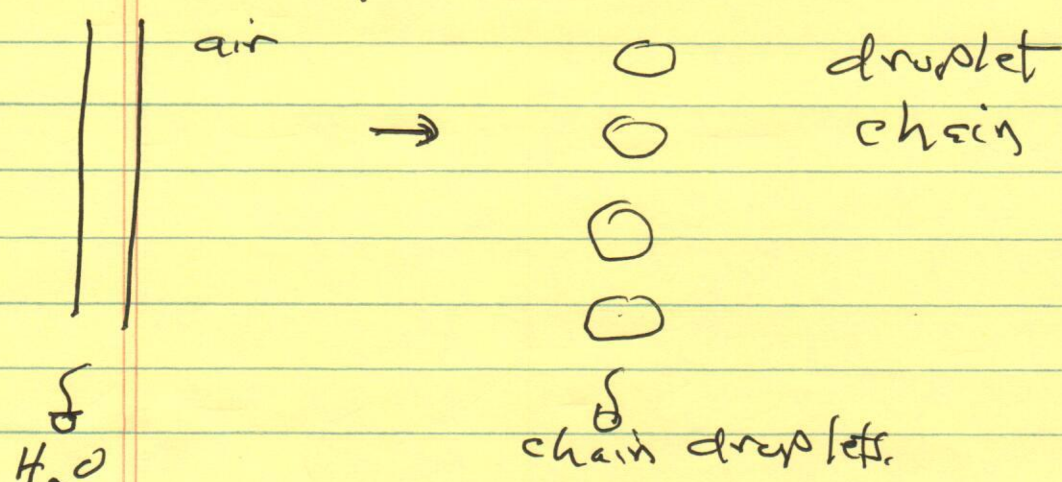
$\downarrow$ capillary wave (short)

$l_{cap} \sim (\sigma/\rho g)^{1/2}$.

in ocean, cross-over at few cm.
Capillarity important at ≤ 5 cm.

N.B.:

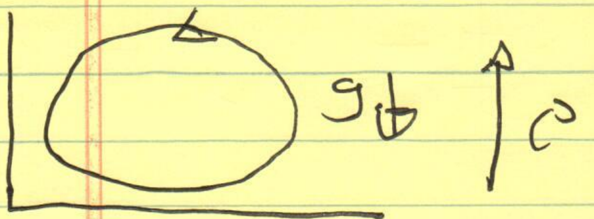
Capillarity (S.T.) can induce instability — classic is line of fluid break-up to string of pearls



skin sausage instability in MHD.

Note also:

Finite layer thickness



$\Rightarrow$ 2D cell — distributed vorticity

$$\omega^2 = -\frac{k_x^2}{k_x^2 + k_z^2}\, g\, \frac{1}{\rho}\frac{\partial \rho}{\partial z}$$

$$= -\frac{k_x^2}{k^2}\, g / L_\rho$$

$$\gamma^2 = \left(\frac{k_x^2}{k^2}\, g / L_\rho\right)^{1/2}$$, so $\gamma \uparrow$ to flat.

— stable stratification:

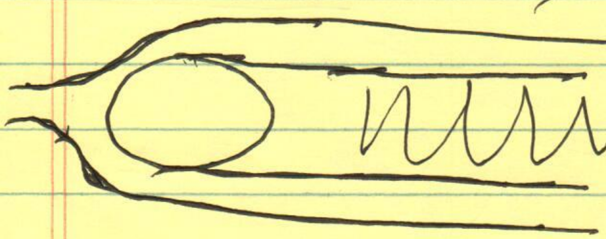
$$\omega^2 = \frac{k_x^2}{k^2}\, g / L_\rho$$ $\rightarrow$ internal wave.

$N^2 \rightarrow$ BV freq.

→ Kelvin – Helmholtz

Recall OV of main story:

outside wake
→ ideal, Bernoulli



← wake, turbulent for $Re > Re_{crit}$

– with no-slip B.C.'s,

$V_n\big|_{surf} = v_T\big|_{surf} = 0$

separation happens, wake forms

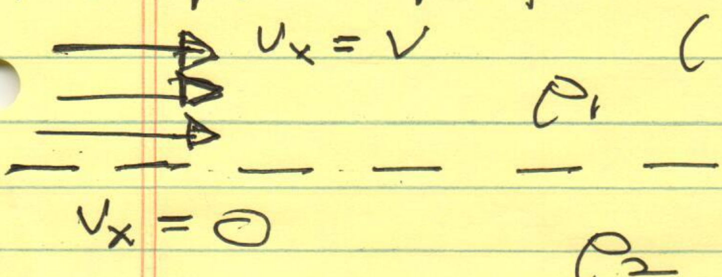
– separation → instability → turbulence.

How?

Instability ⇒ Kelvin – Helmholtz

⇒ free energy → ∇V – flow shear

⇒ simplification: shear layer (interface) idealization



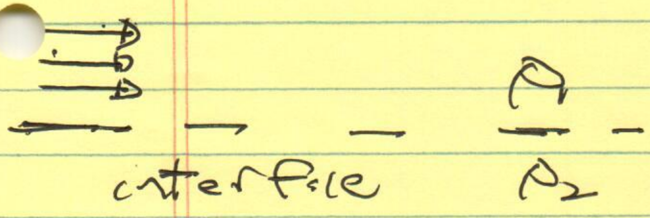
Note:

- $\nabla V = 0$, except interface

- vorticity $\partial V_x / \partial z$ localized to interface

$\Rightarrow$ can play game as' R-T, now with $\underline{V}$, η.

Physical ideas: (1)



high V, low p

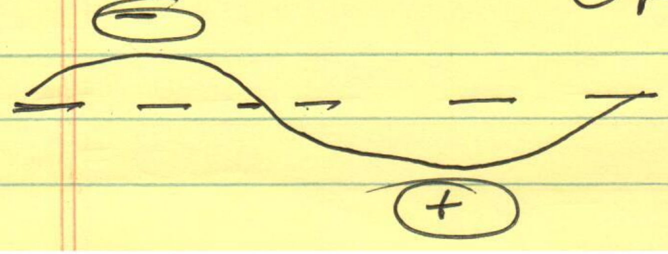
low, zero V, high p

$p + \rho V^2/2 = \text{const}$

(2) δV perturbation
$\rightarrow$ ripple interface

so Bernoulli $\Rightarrow$

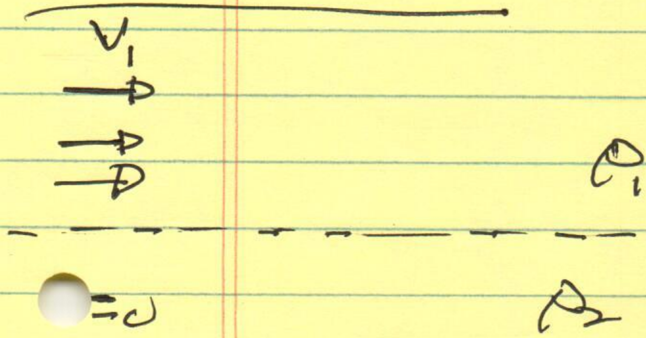
(3) δP



$\delta p < 0 \Rightarrow \delta V > 0$, further
unstable

$\Rightarrow$ KH instability drives viscous mixing
via turbulence, mixing, billows, etc.

To calculate:



$p + \frac{V^2}{2} = \text{const}$
assures eqbrm.

so, as before:

$\underline{\nabla} \cdot \underline{V} = 0$

$\underline{V} = \underline{\nabla}\phi \qquad \underline{\omega} = 0$, except interface

$\nabla^2 \phi = 0$

$\Rightarrow$

wave along interface

$$\phi = \sum_k \phi_k \, e^{ikx} \, e^{-k|z|} \, e^{-i\omega_k t}$$

decays away from interface.

as before;

$\rightarrow \quad \tilde{P}_1(0_+) = \tilde{P}_1(0_-)$

$\rightarrow \quad \eta \underline{\underline{\Rightarrow}}$ interface ripple/displacement

$$\frac{d\eta}{dt} = V_z\Big|_0$$

and $\quad V_z(0_+) = V_z(0_-)$

Now, $\quad \frac{\partial}{\partial t}\underline{V} + \underline{V}\cdot\underline{\nabla}\,\underline{V} = -\frac{\underline{\nabla}P}{\rho}$

$$\rho_{\frac{1}{2}}\left(\frac{\partial}{\partial t}\tilde{V}_{z_{\frac{1}{2}}} + V_{\frac{1}{2}}\,\partial_x \tilde{V}_{z_{\frac{1}{2}}}\right) = (\mp)\nabla_z \tilde{P}_{\frac{1}{2}}$$

$$\tilde{V}_{z_{\frac{1}{2}}} = \frac{\mp i k \tilde{P}_{\frac{1}{2}}}{\rho_{\frac{1}{2}}\left(kV_{\frac{1}{2}} - \omega\right)}$$

and

$\partial_z P \sim$ evanescent

N.B. $\tilde{V}_z$, so i from ∂_t, ∂_x only

$$\frac{d\eta}{dt} = \frac{\partial\tilde{\eta}}{\partial t} + v\frac{\partial\tilde{\eta}}{\partial x} = \tilde{v}_{z_1}$$

$$-i(\omega - kv)\tilde{\eta}_k = \tilde{v}_{z_k}$$

so using Euler/Bernoulli

$$-i\,(\omega - kV)\tilde{\eta}_k = \frac{-ik\tilde{P}_1}{\rho_1\,(kV - \omega)}$$

so

$$\tilde{P}_{1_k} = \frac{-\rho_1\,(kV - \omega)^2}{k}\tilde{\eta}_k$$

$$\tilde{P}_{2_k} = \frac{\rho_2}{k_1}\omega^2\,\tilde{\eta}_k$$ ← note sign. (opposite signs $\tilde{\phi}_1, \tilde{\phi}_2$)

and $\tilde{P}_1 = \tilde{P}_2 \Rightarrow$

$$-\frac{\rho_1}{k}(kv-\omega)^2 = \frac{\rho_2}{k}\omega^2$$

$$\Rightarrow \boxed{\omega = kv\left(\frac{\rho_1 + i(\rho_1\rho_2)^{1/2}}{\rho_1+\rho_2}\right)}$$

so

$$\rightarrow \gamma \sim kv\,\frac{\sqrt{\rho_1\rho_2}}{\rho_1+\rho_2} \qquad \rightarrow \text{KH growth}$$

note $\omega_r \sim kv\left(\rho_1 / (\rho_1+\rho_2)\right)$

no "exchange of stabilities" here.

$\sim \rho_1 = \rho_2 \qquad \gamma \simeq \frac{kv}{2}$

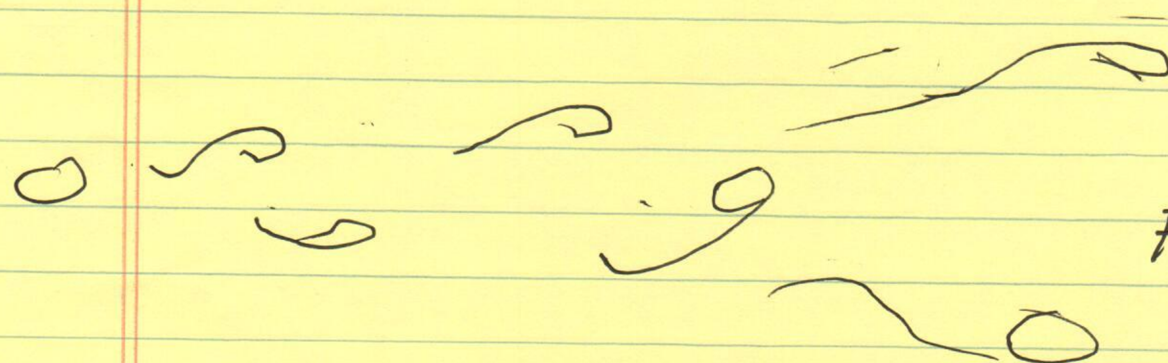
<u>generally</u> $\gamma \sim k(\Delta v)$.

→ what happens?

→ vortex roll-up, billows

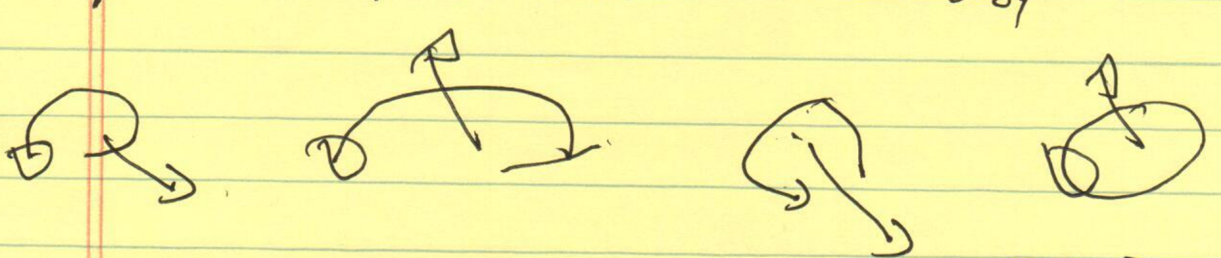
F 2.3, F 2.4

→ Von-Karman vortex street



F1.15

→ n.b. array vortex lines unstable w/r displacements as shown



F2.2

Refs:

→ R-T, K-H:

· S. Chandrasekhar, "Hydrodynamic and Hydromagnetic Stability"

→ K-H: Falkovich

→ Surface Waves, Surface Tension (Laplace Law), KH: Landau/Lifshitz.

→ see also: G. K. Batchelor, "An Introduction to Fluid Mechanics"